Warsaw University of Technology

Faculty of Power and Aeronautical Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

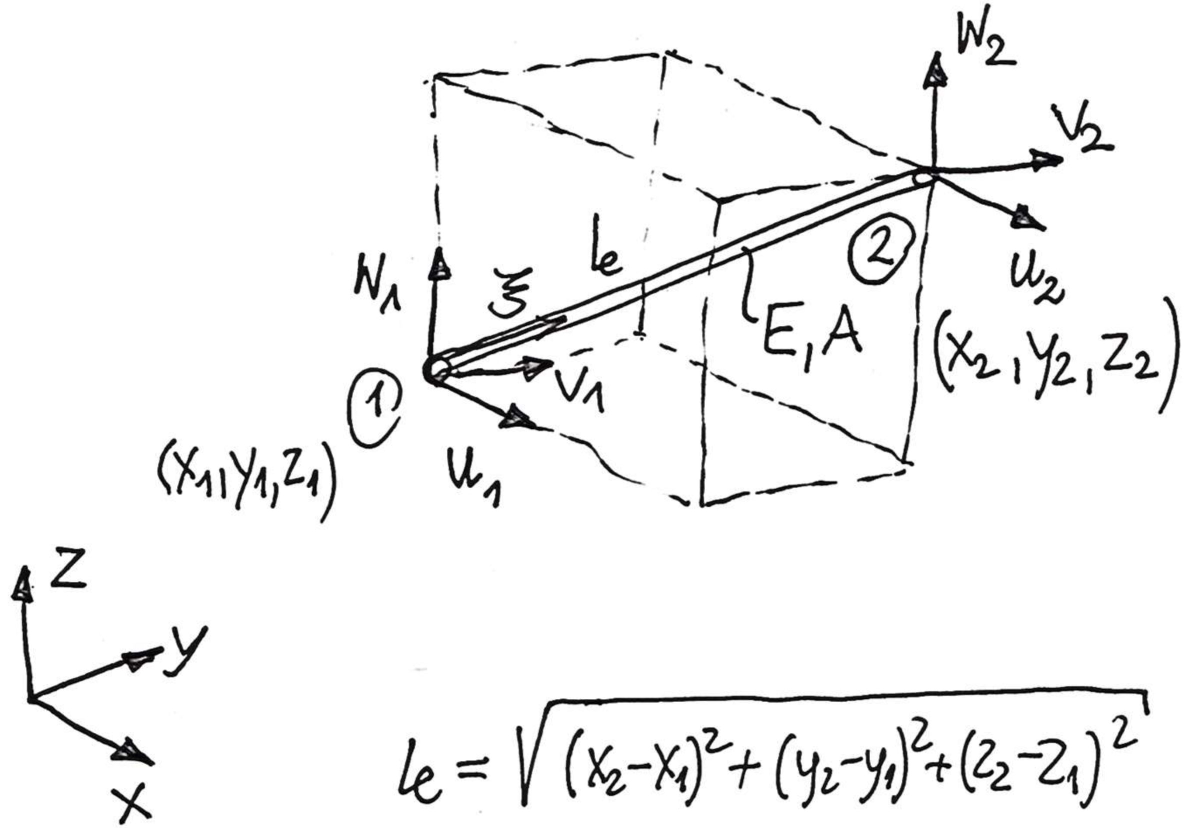
Institute of Aeronautics and Applied Mechanics

Finite element method (FEM)

3D truss finite element

05.2021

3D TRUSS ELEMENT

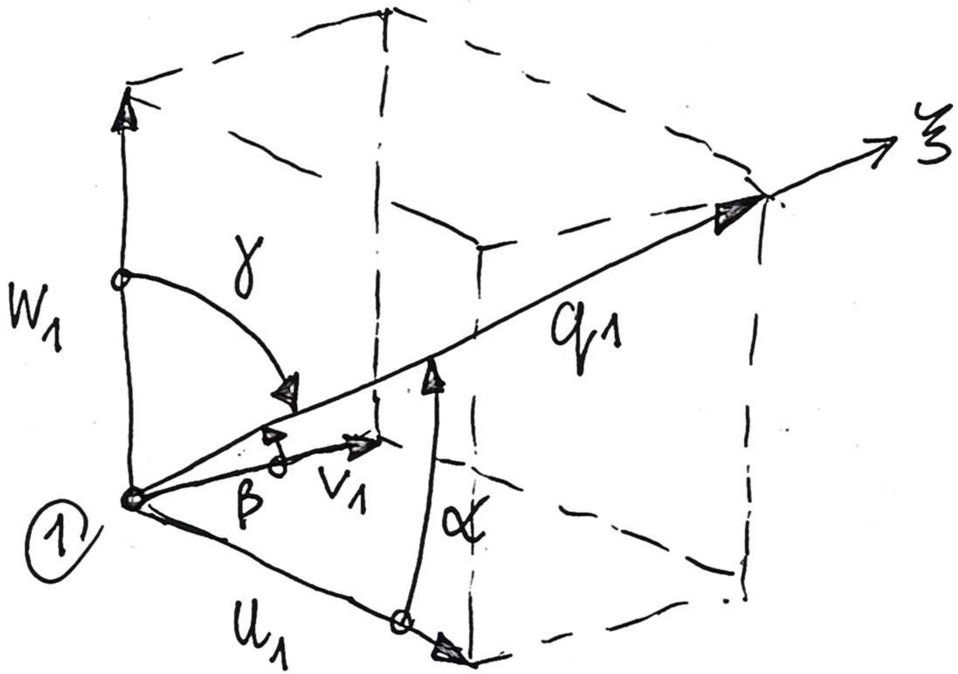


$$\{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e \quad (6\times 1)$$

$$l_e = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

DISPLACEMENT OF NODE ①

AXIAL BAR $\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e$



$$\cos\alpha = \frac{x_2 - x_1}{l_e}$$

$$\cos\beta = \frac{y_2 - y_1}{l_e}$$

$$\cos\gamma = \frac{z_2 - z_1}{l_e}$$

$$u_1 = q_1 \cos\alpha$$
$$v_1 = q_1 \cos\beta$$
$$w_1 = q_1 \cos\gamma$$

$$u_1 \cos\alpha = q_1 \cos^2\alpha$$
$$v_1 \cos\beta = q_1 \cos^2\beta$$
$$w_1 \cos\gamma = q_1 \cos^2\gamma$$

$$u_1 \underbrace{\cos\alpha}_{a} + v_1 \underbrace{\cos\beta}_{b} + w_1 \underbrace{\cos\gamma}_{c} = q_1$$

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$
$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$

$$q_1 = a \cdot u_1 + b \cdot v_1 + c \cdot w_1 + 0 \cdot u_2 + 0 \cdot v_2 + 0 \cdot w_2$$

$$q_2 = 0 \cdot u_1 + 0 \cdot v_1 + 0 \cdot w_1 + a \cdot u_2 + b \cdot v_2 + c \cdot w_2$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \underbrace{\begin{bmatrix} a & b & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \end{bmatrix}} \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}_e$$

$$\underset{2\times 1}{\{q\}_e} = \underset{2\times 6}{[T_t]_e} \cdot \underset{6\times 1}{\{q_g\}_e}$$

$$\underset{1\times 2}{\lfloor q \rfloor_e} = \underset{1\times 6}{\lfloor q_g \rfloor} \cdot \underset{6\times 2}{[T_t]_e^T} \quad ; \quad [T_t]_e^T = \begin{bmatrix} a & 0 \\ b & 0 \\ c & 0 \\ 0 & a \\ 0 & b \\ 0 & c \end{bmatrix}$$

$$U_e = \frac{1}{2} \underset{1\times 2}{\lfloor q \rfloor_e} \cdot \underset{2\times 2}{[K]_e} \cdot \underset{2\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times 6}{\lfloor q_g \rfloor_e} \cdot \underset{6\times 2}{[T_t]_e^T} \underset{2\times 2}{[K]_e} \underset{2\times 6}{[T_t]_e} \cdot \underset{6\times 1}{\{q_g\}_e} =$$

$$= \frac{1}{2} \underset{1\times 6}{\lfloor q_g \rfloor_e} \underset{6\times 6}{[K_g]_e} \cdot \underset{6\times 1}{\{q_g\}_e} \quad , \text{ where:}$$

$$\underset{2\times 2}{[k]_e} = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

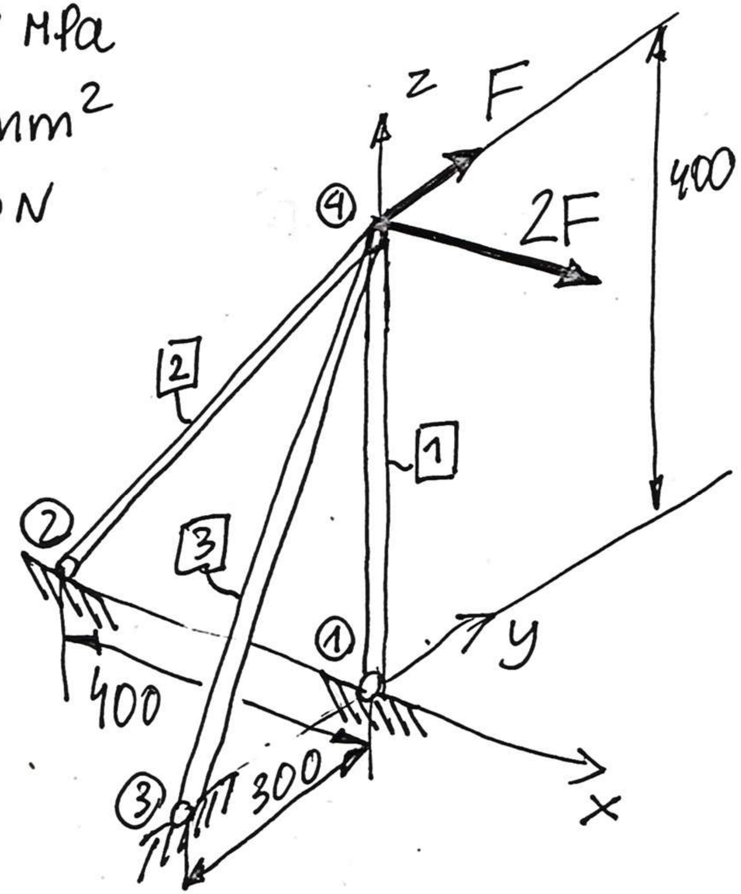
$$\underset{6\times 6}{[k_g]_e} = \frac{EA}{l_e} \begin{bmatrix} a^2 & ab & ac & -a^2 & -ab & -ac \\ ab & b^2 & bc & -ab & -b^2 & -bc \\ ac & bc & c^2 & -ac & -bc & -c^2 \\ -a^2 & -ab & -ac & a^2 & ab & ac \\ -ab & -b^2 & -bc & ab & b^2 & bc \\ -ac & -bc & -c^2 & ac & bc & c^2 \end{bmatrix}$$

EXAMPLE. BUILD A FINITE ELEMENT MODEL OF A 3D TRUSS. FIND NODAL DISPLACEMENTS, STRESS, INTERNAL FORCES AND REACTIONS.

$E = 2 \cdot 10^5$ MPa

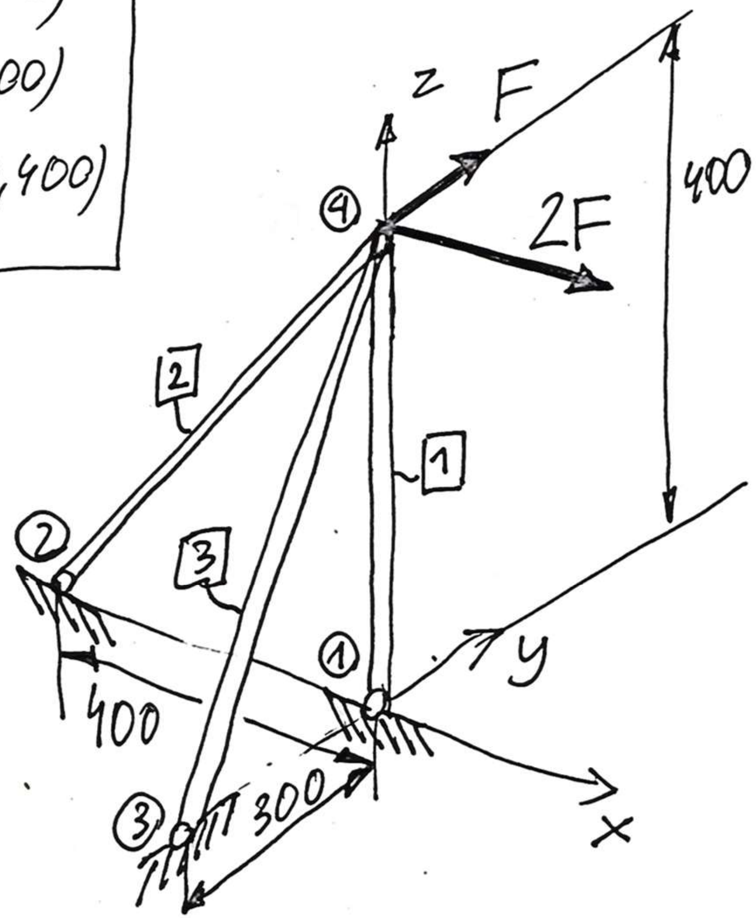
$A = 100 \text{mm}^2$

$F = 1500 N$



$$\{q\}_{12\times1} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

ELEMENT	NODES
[1]	(1) (0, 0, 0) → (4) (0, 0, 400)
[2]	(2) (-400, 0, 0) → (4) (0, 0, 400)
[3]	(3) (0, -300, 0) → (4) (0, 0, 400)

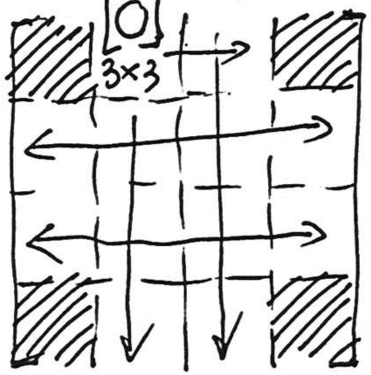


ELEMENT [1] ; $l_1 = 400\,mm$

$$a_1 = \frac{0-0}{l_1} = 0 \;;\quad b_1 = \frac{0-0}{l_1} = 0, \quad c_1 = \frac{400-0}{400} = 1$$

$$\underset{2\times6}{[T_t]_1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{6\times6}{[K_g]_1} = \frac{EA}{l_1}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \;;\quad \underset{12\times12}{[K_g]_1^*} =$$



ELEMENT $\boxed{2}$

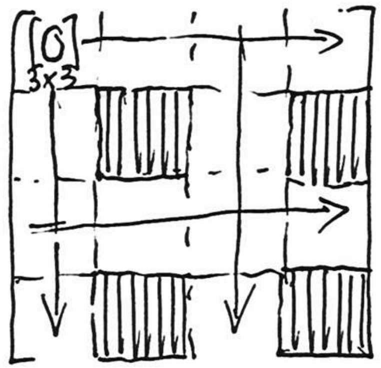
$$l_2 = \sqrt{(0-(-400))^2 + (0-0)^2 + (400-0)^2} = 400\sqrt{2}\ mm$$

$$a_2 = \frac{0-(-400)}{400\sqrt{2}} = \frac{\sqrt{2}}{2}\ ,\quad b_2 = \frac{0-0}{400\sqrt{2}} = 0\ ,\quad c_2 = \frac{400-0}{400\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\underset{2\times 6}{[T_t]_2} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\underset{6\times 6}{[k_g]_2} = \frac{EA}{l_2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\underset{12\times 12}{[k_g]_2^*} =$$



ELEMENT

3

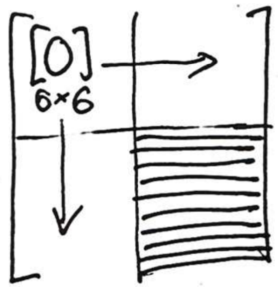
$$l_3 = \sqrt{(0-0)^2 + (0-(-300))^2 + (400-0)^2} = 500\,mm$$

$$a_3 = \frac{0-0}{500} = 0 \quad , \quad b_3 = \frac{0-(-300)}{500} = \frac{3}{5} = 0.6 \quad , \quad c_3 = \frac{400-0}{500} = 0.8$$

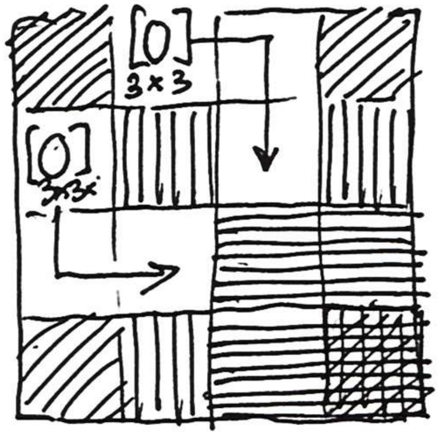
$$\underset{2\times 6}{[T_t]_3} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix}$$

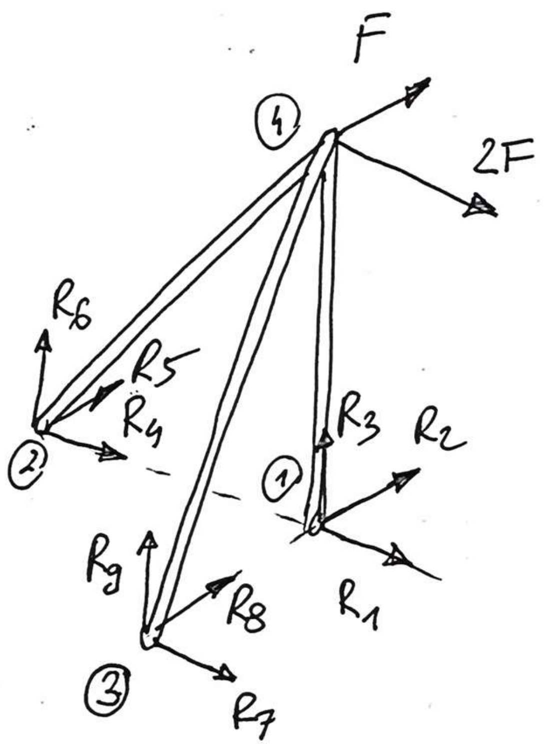
$$\underset{6\times 6}{[k_g]_3} = \frac{EA}{l_3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.36 & 0.48 & 0 & -0.36 & -0.48 \\ 0 & 0.48 & 0.64 & 0 & -0.48 & -0.64 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.36 & -0.48 & 0 & 0.36 & 0.48 \\ 0 & -0.48 & -0.64 & 0 & 0.48 & 0.64 \end{bmatrix}$$

$$\underset{12\times 12}{[k_g]_3^*} =$$



$$\underset{12\times 12}{[K]} = [k_g]_1^* + [k_g]_2^* + [k_g]_3^* =$$



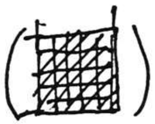


$$\{F\}_{12\times1} = \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \\ 2F \\ F \\ 0 \end{Bmatrix}$$

BOUNDARY CONDITIONS

$$u_1 = v_1 = w_1 = u_2 = v_2 = w_2 = u_3 = v_3 = w_3 = 0$$

$$N = NDOF - NOF = 12 - 9 = 3$$



$$\underset{3\times3}{[K]} \cdot \begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 2F \\ F \\ 0 \end{Bmatrix}$$

$$[K]_{3\times3} = EA \begin{bmatrix} \frac{1}{2l_2} & 0 & \frac{1}{2l_2} \\ 0 & \frac{0.36}{l_3} & \frac{0.48}{l_3} \\ \frac{1}{2l_2} & \frac{0.48}{l_3} & \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) \end{bmatrix}$$

$$[K]_{3\times3} \cdot \begin{Bmatrix} u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 2F \\ F \\ 0 \end{Bmatrix}$$

$$\begin{cases} \frac{1}{2l_2} \cdot u_4 + 0 \cdot v_4 + \frac{1}{2l_2} w_4 = \frac{2F}{EA} & \Big| \cdot 2l_2 \\ 0 \cdot u_4 + \frac{0.36}{l_3} v_4 + \frac{0.48}{l_3} w_4 = \frac{F}{EA} & \Big| \cdot \frac{100\, l_3}{36} \\ \frac{1}{2l_2} \cdot u_4 + \frac{0.48}{l_3} v_4 + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right) w_4 = 0 \end{cases}$$

$$\begin{cases} u_4 + w_4 = \frac{4Fl_2}{EA} & \Rightarrow u_4 = \frac{4Fl_2}{EA} - w_4 \\ v_4 + \frac{4}{3}w_4 = \frac{100\,Fl_3}{36\,EA} & \Rightarrow v_4 = \frac{100\,Fl_3}{36\,EA} - \frac{4}{3}w_4 \\ \frac{1}{2l_2}u_4 + \frac{0.48}{l_3}v_4 + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right)w_4 = 0 \end{cases}$$

$$\frac{1}{2l_2}\cdot\left(\frac{4Fl_2}{EA} - w_4\right) + \frac{0.48}{l_3}\cdot\left(\frac{100}{36}\frac{Fl_3}{EA} - \frac{4}{3}w_4\right) + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3}\right)w_4 = 0$$

$$\frac{2F}{EA} + \frac{4F}{3EA} + \left(\frac{1}{l_1} + \frac{1}{2l_2} + \frac{0.64}{l_3} - \frac{1}{2l_2} - \frac{0.64}{l_3}\right)w_4 = 0$$

$$w_4 = -\left(\frac{2F}{EA} + \frac{4F}{3EA}\right)l_1 = -\frac{10\,Fl_1}{3\,EA} = -0.1\,mm$$

$$u_4 = \frac{4Fl_2}{EA} + \frac{10\,Fl_1}{3EA} = \frac{12\,Fl_2}{3EA} + \frac{10\,Fl_1}{3EA} = \frac{(12l_2 + 10l_1)F}{3EA} = 0.27\,mm$$

$$v_4 = \frac{100\,Fl_3}{36\,EA} - \frac{4}{3}\cdot\left(-\frac{10\,Fl_1}{3\,EA}\right) = \frac{100Fl_3}{36EA} + \frac{40\,Fl_1}{9\,EA} = \frac{(100l_3 + 160l_1)F}{36EA} = 0.2375\,mm$$

REACTIONS :

$$[K]_{12\times12}\cdot\{q\}_{12\times1}=\{F\}_{12\times1}$$

0	0	0	0	0	0	0	0	0	0	0	0		0		0
0	0	0	0	0	0	0	0	0	0	0	0		0		0
0	0	50000	0	0	0	0	0	0	0	0	-50000		0		5000
0	0	0	17677.67	0	17677.67	0	0	0	-17677.7	0	-17677.7		0		-3000
0	0	0	0	0	0	0	0	0	0	0	0		0		0
0	0	0	17677.67	0	17677.67	0	0	0	-17677.7	0	-17677.7	·	0	=	-3000
0	0	0	0	0	0	0	0	0	0	0	0		0		0
0	0	0	0	0	0	0	14400	19200	0	-14400	-19200		0		-1500
0	0	0	0	0	0	0	19200	25600	0	-19200	-25600		0		-2000
0	0	0	-17677.7	0	-17677.7	0	0	0	17677.67	0	17677.67		0.269706		3000
0	0	0	0	0	0	0	-14400	-19200	0	14400	19200		0.2375		1500
0	0	-50000	-17677.7	0	-17677.7	0	-19200	-25600	17677.67	19200	93277.67		-0.1		0

$$R_1=0\ ,\quad R_2=0\ ,$$

$$R_3=-\frac{EA}{l_1}\cdot w_4=-\frac{EA}{l_1}\cdot\left(-\frac{10}{3}\frac{Fl_1}{EA}\right)=\frac{10}{3}F=5000N$$

$$R_4=-\frac{EA}{2l_2}\cdot u_4-\frac{EA}{2l_2}w_4=-\frac{EAF}{2l_2}\left(\frac{12l_2+10l_1}{3EA}-\frac{10l_1}{3EA}\right)=-2F=-3000N$$

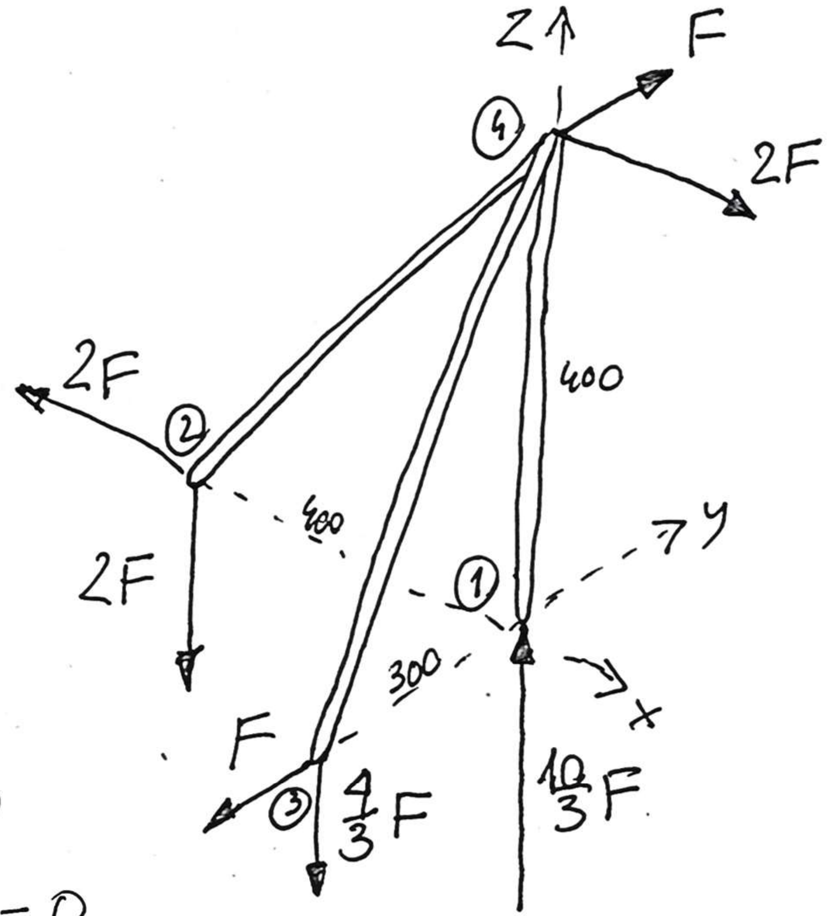
$$R_5=0\ ,\quad R_6=-\frac{EA}{2l_2}u_4-\frac{EA}{2l_2}w_4=R_4=-2F=-3000N$$

$$R_7=0\ ,\quad R_8=-\frac{0.36EA}{l_3}\cdot v_4-\frac{0.48EA}{l_3}w_4=$$

$$=-\frac{0.36EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA}+\frac{0.48EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA}=-F=-1500N$$

$$R_9=\frac{-0.48EA}{l_3}\cdot v_4-\frac{0.64EA}{l_3}w_4=$$

$$=-\frac{0.48EA}{l_3}\cdot\frac{100(l_3+1.6l_1)F}{36EA}+\frac{0.64\cdot EA}{l_3}\cdot\frac{10}{3}\frac{Fl_1}{EA}=-\frac{4F}{3}=-2000N$$



$$\sum F_x = 0 : \quad -2F + 2F = 0$$

$$\sum F_y = 0 : \quad -F + F = 0$$

$$\sum F_z = 0 : \quad \frac{10}{3}F - 2F - \frac{4}{3}F = 0$$

$$\sum M_x^{①} = 0 : \quad \frac{4}{3}F \cdot 300\,mm - F \cdot 400\,mm = 0$$

$$\sum M_y^{①} = 0 : \quad -2F \cdot 400\,mm + 2F \cdot 400\,mm = 0$$

$$\sum M_z^{②} = 0 : \quad -F \cdot 400\,mm + F \cdot 400\,mm = 0$$

ELEMENT SOLUTION

[1]

$$\{q\}_1^{2\times L} = [T_t]_1 \cdot \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ w_4 \end{Bmatrix}$$

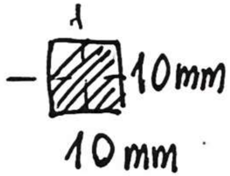
$$\varepsilon_1 = \frac{w_4 - 0}{l_1} = -\frac{10\,F}{3\,EA} = -0.25 \cdot 10^{-3}$$

$$\sigma_1 = E \cdot \varepsilon_1 = -\frac{10}{3}\frac{F}{A} = -50\,MPa \qquad N_1 = \sigma_1 \cdot A = -\frac{10}{3}F = -5000\,N$$

(COMPRESSION – POSSIBLE BUCKLING)

EULER FORCE

$$F_{CR} = \frac{\pi^2 E J}{L_1^2}$$



$$J = \frac{(10mm)^4}{12}$$

$$F_{CR} = \frac{\pi^2 E \cdot 10^4 mm^4}{12 \cdot 400^2 mm^2} = \frac{\pi^2 \cdot 2 \cdot 10^5 \cdot 10^4 \, Nmm^2}{12 \cdot 16 \cdot 10^4 mm^2} = 10281 N$$

SAFETY FACTOR $n = 2$

$$|N_1| < \frac{F_{CR}}{n}$$

[2]

$$\{q\}_2 = \underset{2\times 6}{[\Gamma_t]_2} \cdot \begin{Bmatrix} u_2 \\ v_2 \\ w_2 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{\sqrt{2}}{2} \cdot (u_4 + w_4) \end{Bmatrix}$$

$$\varepsilon_2 = \frac{\sqrt{2}}{2l_2}(u_4 + w_4) = \frac{\sqrt{2}}{2l_2} \cdot \left(\frac{(12 l_2 + 10 l_1) F}{3EA} - \frac{10 F l_1}{3EA} \right) = \frac{2\sqrt{2}\, F}{EA} = 0.212 \cdot 10^{-3}$$

$$\sigma_2 = E \varepsilon_2 = 42.43 \text{ MPa},$$

$$N_2 = \sigma_2 \cdot A = 4243 N$$

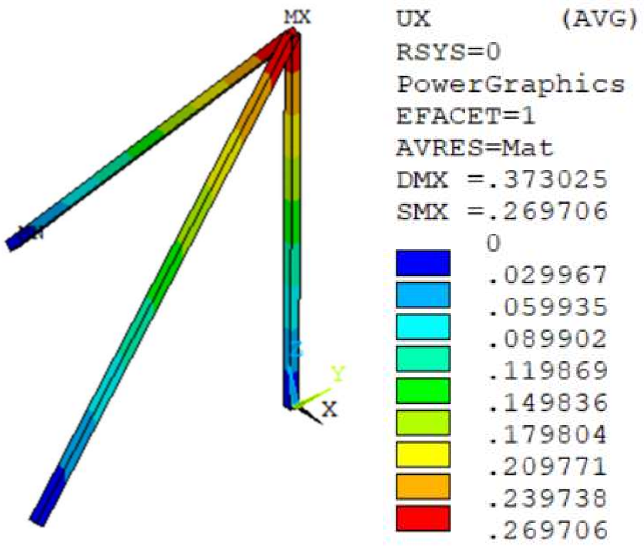
3

$$\{q\}_3 = [T_t]_3 \cdot \begin{Bmatrix} u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.8 \end{bmatrix} \cdot \begin{Bmatrix} 0 \\ 0 \\ 0 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.6 \cdot v_4 + 0.8 \cdot w_4 \end{Bmatrix}$$

$$\varepsilon_3 = \frac{1}{l_3}\left(0.6 \cdot \frac{(100 l_3 + 160 l_1)F}{36 EA} - 0.8 \cdot \frac{10}{3}\frac{F l_1}{EA}\right) = \frac{5F}{3EA} = 0.125 \cdot 10^{-3}$$

$$\sigma_3 = E \cdot \varepsilon_3 = 25 \, MPa$$

$$N_3 = \sigma_3 \cdot A = 2500 N$$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PowerGraphics
EFACET=1
AVRES=Mat
DMX =.373025
SMX =.269706
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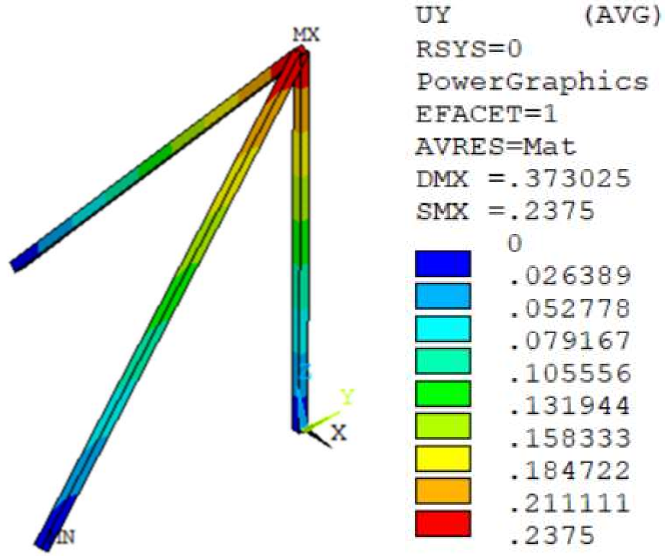
.059935
.089902
.119869
.149836
.179804
.209771
.239738
.269706
Y
X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UY (AVG)
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PowerGraphics
EFACET=1
AVRES=Mat
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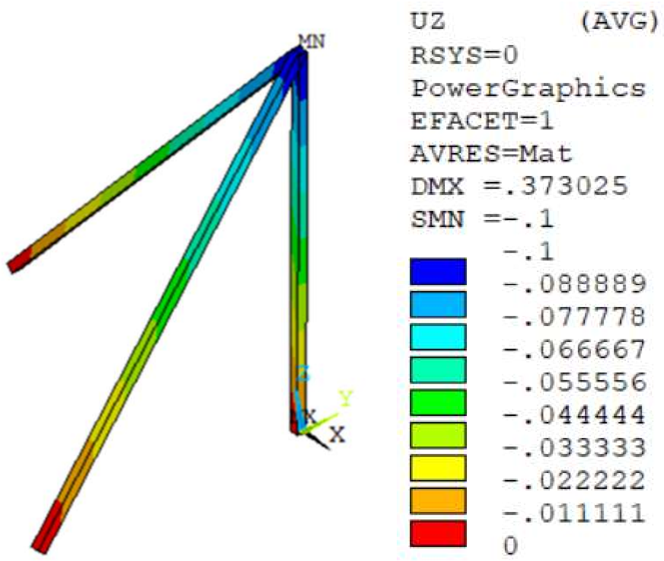
.184722
.211111
.2375
Y
X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-.066667
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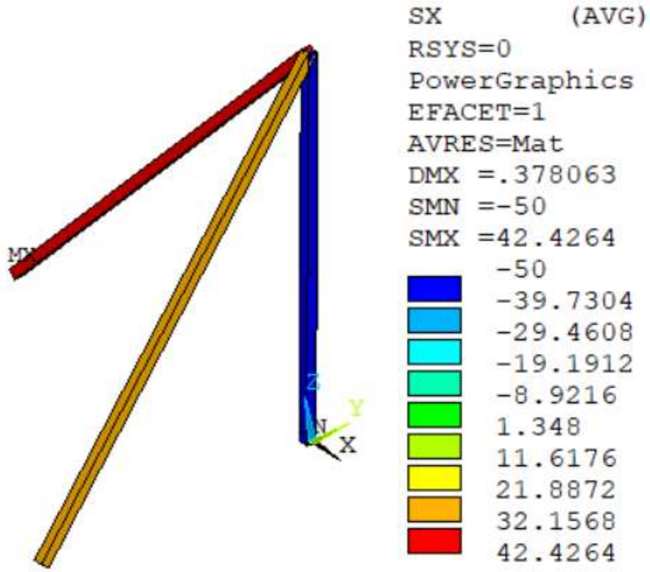
-.033333
-.022222
-.011111
0
Y
X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AVRES=Mat
DMX =.378063
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-8.9216
1.348
11.6176
21.8872
32.1568
42.4264
Y
X